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A simple analytic approximation to the Rayleigh-Bénard stability threshold

Andrea Prosperetti[a)]

Department of Mechanical Engineering, Johns Hopkins University, Baltimore, Maryland 21218, USA

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The Rayleigh-Bénard linear stability problem is solved by means of a Fourier series expansion. It is found that truncating the series to just the first term gives an excellent explicit approximation to the marginal stability relation between the Rayleigh number and the wave number of the perturbation. Where the error can be compared with published exact results, it is found not to exceed a few percent over the entire wave number range. Several cases with no-slip boundaries of equal or unequal thermal conductivities are considered explicitly. © *2011 American Institute of Physics*. [doi:10.1063/1.3662466]

I. INTRODUCTION

A fluid layer heated from below—the so-called Rayleigh-Bénard problem—is one of the most classical examples of fluid dynamic stability problems (see, e.g., Refs. 1 and 2). In view of its widespread occurrence in a large number of scientific disciplines it has formed the object of literally thousands of investigations since the original studies of Bénard in 1900 and Lord Rayleigh in 1916.[3,4] It is, therefore, rather surprising that the simple approximate solution of the linear threshold problem described in this paper does not appear to be in the literature. As an example, in the particularly important case of no-slip plates with fixed temperatures, we find that the relation

$$Ra = \frac{(\pi^2 + k^2H^2)^3}{k^2H^2}\left[1 - \frac{16\pi^2 kH}{(\pi^2+k^2H^2)^2}\frac{\cosh^2(kH)/2}{\sinh(kH)+kH}\right]^{-1}, \tag{1}$$

in which Ra is the Rayleigh number, H is the separation of the plates, and k is the dimensional wave number, approximates the exact result to better than 0.5% over the entire wave number range.

The relative simplicity of the results given by our approach enables us to investigate several other cases and in particular the effect of finite thermal conductivity of the plates as well as plates of different conductivities.[5,6] When exact results are available, we find that our approximations have an error of at most a few percent.

To be sure, most of the interest in the Rayleigh-Bénard problem centers on non-linear effects (see, e.g., the reviews provided in Refs. 7–10) rather than the linear problem studied here. Nevertheless, the availability of a simple approximate solution may be used to develop analytically tractable weakly non-linear theories, to check numerical codes and as a starting point for numerical simulations. Furthermore, the method of analysis used here is fairly general and may be applicable to other stability problems.

II. MATHEMATICAL FORMULATION

We consider a Boussinesq fluid contained between two infinitely extended plates normal to the direction of gravity separated by a distance H. Since the mathematical framework of the linear problem is so well known (see, e.g., Refs. 1 and 2), we can omit many details.

In the equilibrium state, the fluid is at rest with a temperature field $T_0(z)$ given by

$$T_0 = \frac{1}{2}(T_C + T_H) - \frac{z}{H}(T_H - T_C), \tag{2}$$

in which $T_{H,C}$ denote the temperatures of the hotter lower plate and of the colder upper plate. The frame of reference is chosen, so that the two plates are located at $z = \pm\frac{1}{2}H$. The temperature gradient in the fluid is therefore constant and is given by

$$G \equiv -\frac{dT_0}{dz} = \frac{T_H - T_C}{H}. \tag{3}$$

There will also be a steady temperature field in the upper plate (index u), given by

$$T_{0u} = \frac{1}{2}(T_C + T_H) - \frac{1}{2}HG - \frac{K}{K_u}G\left(z - \frac{1}{2}H\right), \tag{4}$$

and in the lower plate (index l), given by

$$T_{0l} = \frac{1}{2}(T_C + T_H) + \frac{1}{2}HG - \frac{K}{K_l}G\left(z + \frac{1}{2}H\right). \tag{5}$$

Here, K is the thermal conductivity of the fluid and $K_{l,u}$ those of the plates, not necessarily equal. These expressions embody the continuity of temperature and heat fluxes at the plate surfaces exposed to the fluid. For simplicity, in this paper, we treat the plates as semi-infinite. The results of Ref. 6 show that this is a good approximation for plates with a finite thickness of the order of H or greater.

Perturbed quantities are denoted by a prime and are determined from the linearized form of the equations expressing conservation mass, momentum, and energy:

a)Also at Faculty of Science and Technology, Impact Institute and J. M. Burgers Center for Fluid Dynamics, University of Twente, 7500AE Enschede, The Netherlands. Electronic mail: prosperetti@jhu.edu.

$$\nabla \cdot \mathbf{u}' = 0, \tag{6}$$

$$\frac{\partial \mathbf{u}'}{\partial t} = -\frac{1}{\rho}\nabla p' + \nu\nabla^2\mathbf{u}' - \beta T'\mathbf{g}, \tag{7}$$

$$\frac{\partial T'}{\partial t} + \mathbf{u}' \cdot \nabla T_0 = D\nabla^2 T', \tag{8}$$

$$\frac{\partial T'_{l,u}}{\partial t} = D_{l,u}\nabla^2 T'_{l,u}. \tag{9}$$

Here, $\mathbf{u}'$, p', and T' are the fluid velocity, pressure, and temperature; $\mathbf{g}$ is the acceleration of gravity, ρ is the density, ν is the kinematic viscosity, β is the thermal expansion coefficient, and D is the thermal diffusivity. The perturbation temperatures and thermal diffusivities of the plates are denoted by $T'_{l,u}$ and $D_{l,u}$.

As in the standard theory (see Refs. 1 and 2), it is convenient to take the double curl of the momentum Eq. (7) and to project the result on the vertical direction to find

$$\frac{\partial}{\partial t}\nabla^2 u'_z = \nu\nabla^2(\nabla^2 u'_z) + \beta g \nabla_2^2 T', \tag{10}$$

in which u'_z is the vertical velocity and ∇_2^2 is the Laplacian operator in the horizontal planes.

We now assume that all the perturbation quantities are proportional to a function of z multiplied by $e^{st}f(x,y)$, where s is a constant and the function f satisfies

$$\nabla_2^2 f = -k^2 f, \tag{11}$$

with k the wave number of the perturbation. Before writing down the result we also effect a non-dimensionalization using H, H^2/ν, and $T_H - T_C$ as fundamental length, time, and temperature scales, respectively. With this further step, and upon retaining the same symbols to denote the non-dimensional variables, the equations become

$$s\left(\frac{\partial^2}{\partial z^2} - k^2\right)u'_z = \left(\frac{\partial^2}{\partial z^2} - k^2\right)^2 u'_z - \frac{Ra}{Pr}k^2 T', \tag{12}$$

$$sT' - u'_z = \frac{1}{Pr}\left(\frac{\partial^2}{\partial z^2} - k^2\right)T', \tag{13}$$

$$sT'_{l,u} = \frac{D_{l,u}}{\nu}\left(\frac{\partial^2}{\partial z^2} - k^2\right)T'_{l,u}, \tag{14}$$

in which $Pr = \nu/D$ is the Prandtl number and

$$Ra = \frac{\beta g H^3 (T_H - T_C)}{\nu D} \tag{15}$$

is the Rayleigh number, the fundamental parameter of the problem.

At the plate surfaces, we impose continuity of temperature and heat flux and zero normal velocity. If no-slip conditions prevail, the tangential velocity vanishes as well so that, from the equation of continuity, the first derivative of u'_z also vanishes. In the case of free-slip conditions, from the vanishing of the tangential stress, it is the second derivative of u'_z which vanishes. We do not consider this latter situation as the standard theory already produces a very simple result.[1]

III. MARGINAL STABILITY CONDITIONS

It is readily shown proceeding in the standard way that the differential operator in the system (10) to (14) is self-adjoint with the boundary conditions stated, so that the eigenvalues s are real. In particular, at marginal stability conditions, s vanishes and the system reduces to

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2 u'_z - \frac{Ra}{Pr}k^2 T' = 0, \tag{16}$$

$$\frac{1}{Pr}\left(\frac{\partial^2}{\partial z^2} - k^2\right)T' + u'_z = 0, \tag{17}$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)T'_{l,u} = 0. \tag{18}$$

Upon solving Eq. (16) for T', we find

$$T' = \frac{Pr}{Ra\,k^2}\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2 u'_z, \tag{19}$$

which can be substituted into Eq. (17) with the result

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)^3 u'_z = -Ra\,k^2 u'_z. \tag{20}$$

The energy equations in the plates are readily solved to find

$$T'_l = T'_H e^{k(z+1/2)}, \qquad T'_u = T'_C e^{-k(z-1/2)}, \tag{21}$$

in which $T'_{H,C}$ denote the temperature perturbations at the hot and cold plates. Upon imposing the continuity of heat fluxes at the plates $z = \pm 1/2$, we then have

$$k\kappa_l T'_H - \left.\frac{\partial T'}{\partial z}\right|_{-1/2} = 0, \qquad k\kappa_u T'_C + \left.\frac{\partial T'}{\partial z}\right|_{1/2} = 0,$$
$$\text{with} \quad \kappa_{l,u} = \frac{K_{l,u}}{K}, \tag{22}$$

which, by means of Eq. (19), become additional boundary conditions on the perturbation velocity.

IV. SOLUTION

The general solution of the 6th-order differential equation (20) is readily written down. Upon imposing the boundary conditions one is then led to an equation relating the wave number k and the Rayleigh number Ra along the marginal stability boundary (see, e.g., Refs. 1 and 2). The procedure is straightforward, but the resulting equation is too involved to obtain explicit results. Here, we follow a different path.

A complete set of eigenfunctions of the operator $\partial^2/\partial z^2$ vanishing at $z = \pm 1/2$ is given by

$$\{\sin 2n\pi z\}, \qquad \{\cos(2n-1)\pi z\}, \qquad n = 1, 2, \ldots. \tag{23}$$

Since $u_z' = 0$ on the plates, it can be expanded on the basis formed by these eigenfunctions

$$u_z' = -\frac{1}{k^2 Ra}\sum_{n=1}^{\infty}\Big\{[k^2+4\pi^2 n^2]^3 U_n \sin 2n\pi z + [k^2+\pi^2(2n-1)^2]^3 \tilde{U}_n \cos(2n-1)\pi z\Big\}. \tag{24}$$

Here, the as yet undetermined coefficients of the expansion have been written so as to simplify later expressions. Upon substituting into the right-hand side of Eq. (20), we have

$$(\partial^2 - k^2)^3 u_z' = \sum_{n=1}^{\infty}\Big\{[k^2+4\pi^2 n^2]^3 U_n \sin 2n\pi z + [k^2+\pi^2(2n-1)^2]^3 \tilde{U}_n \cos(2n-1)\pi z\Big\}, \tag{25}$$

in which we write ∂ in place of $\partial/\partial z$. This relation may be regarded as the Fourier expansion of the quantity in the left-hand side. We integrate once to find

$$(\partial^2 - k^2)^2 u_z' = -\sum_{n=1}^{\infty}\Big\{[k^2+4\pi^2 n^2]^2 U_n \sin 2n\pi z + [k^2+\pi^2(2n-1)^2]^2 \tilde{U}_n \cos(2n-1)\pi z\Big\} + W_c F_c + W_s F_s, \tag{26}$$

in which, for later convenience, the solutions of the homogeneous equation have been written as

$$F_c = \cosh kz, \qquad F_s = \sinh kz, \tag{27}$$

and $W_{c,s}$ are integration constants. Two further integrations give

$$u_z' = -\sum_{n=1}^{\infty}\big(U_n \sin 2n\pi z + \tilde{U}_n \cos(2n-1)\pi z\big) + W_c H_c + W_s H_s + V_c G_c + V_s G_s + U_c F_c + U_s F_s, \tag{28}$$

in which $V_{c,s}$ and $U_{c,s}$ are integration constants, and the terms in the second line are additional four solutions of the homogeneous equation associated with the left-hand side of Eq. (20). It is convenient to express them in the form

$$G_{c,s} = \frac{1}{2k}\partial_k F_{c,s}, \quad H_{c,s} = \frac{1}{4k}\partial_k G_{c,s} = \frac{1}{4k}\partial_k\left(\frac{z}{2k}F_{s,c}\right). \tag{29}$$

Equating the two representations (24) and (28) of u_z', multiplying by each eigenfunction in turn and integrating between $z = -1/2$ and $z = 1/2$, we find

$$-\frac{[k^2+4\pi^2 n^2]^3}{2Ra\,k^2} U_n = -\frac{1}{2}U_n + W_s I_n^{H,s} + V_s I_n^{G,s} + U_s I_n^{F,s}, \tag{30}$$

$$-\frac{[k^2+(2n-1)^2\pi^2]^3}{2Ra\,k^2}\tilde{U}_n = -\frac{1}{2}\tilde{U}_n + W_c I_n^{H,c} + V_c I_n^{G,c} + U_c I_n^{F,c}, \tag{31}$$

in which

$$I_n^{F,s} = \int_{-1/2}^{1/2} \sin 2n\pi z\, F_s dz, \qquad I_n^{F,c} = \int_{-1/2}^{1/2} \cos(2n-1)\pi z\, F_c dz; \tag{32}$$

$I_n^{G,c,s}$, $I_n^{H,c,s}$ are given by similar expressions with $F_{c,s}$ replaced by $G_{c,s}$ and $H_{c,s}$, respectively.

The boundary conditions permit now the determination of the integration constants $U_{c,s}, V_{c,s}, W_{c,s}$. The algebra is tedious but straightforward, and it is carried out in detail in the supplementary material.[15]

It is found that, from the velocity boundary conditions, the $U_{c,s}$ and $V_{c,s}$ can be expressed in terms of the $W_{c,s}$; the relevant expressions are given in the Appendix. Finally, the temperature boundary conditions determine $W_{c,s}$. When these results are substituted into Eqs. (30) and (31), we find

$$\left(1 - \frac{[(2m-1)^2\pi^2+k^2]^3}{Ra k^2}\right)\tilde{U}_m = \alpha_m^c \sum_{n=1}^{\infty}(2n-1)(-1)^{n+1}\tilde{U}_n + \beta_m^c \sum_{n=1}^{\infty}(-1)^{n+1}[\nu_n^c U_n + \mu_n^c \tilde{U}_n], \tag{33}$$

$$\left(1 - \frac{(4\pi^2 m^2+k^2)^3}{Ra\,k^2}\right)U_m = \alpha_m^s \sum_{n=1}^{\infty} 2n(-1)^{n+1} U_n - \beta_m^s \sum_{n=1}^{\infty}(-1)^{n+1}[\mu_n^s U_n + \nu_n^s \tilde{U}_n], \tag{34}$$

with the quantities $\alpha_m^{c,s}$, $\beta_m^{c,s}$, $\mu_m^{c,s}$, and $\nu_m^{c,s}$ dependent on m and k given in the Appendix. By way of example, we show here explicit expressions for the quantities appearing in the first equation for $n = 1$

$$\alpha_1^c = \frac{16\pi^2 k}{(\pi^2+k^2)^2}\frac{\cosh^2 k/2}{\sinh k + k}, \tag{35}$$

$$\beta_1^c = \frac{\pi}{k^2(\pi^2+k^2)^2}\left[\frac{k^2\sinh k/2 + (\sinh k - k)\cosh k/2}{\sinh k + k} - \frac{4k^2\cosh k/2}{\pi^2+k^2}\right], \tag{36}$$

$$\mu_1^c = \pi(\pi^2+k^2)^2 \frac{(\kappa_l+\kappa_u)\sinh k/2 + 2\cosh k/2}{k[(\kappa_l\kappa_u+1)\sinh k + (\kappa_l+\kappa_u)\cosh k]}, \quad (37)$$

$$\nu_1^c = \frac{2\pi(4\pi^2+k^2)^2 \sinh k/2}{k[(\kappa_l\kappa_u+1)\sinh k + (\kappa_l+\kappa_u)\cosh k]}(\kappa_l-\kappa_u). \quad (38)$$

It is seen from Eq. (38) and, more generally, from the definitions (A9) and (A10), that $\nu_m^{c,s} \propto (\kappa_l - \kappa_u)$ so that these quantities vanish when the plates have equal conductivities. In this case, the two equation sets decouple and the modes separate into even and odd families, as is well known. It also follows from this dependence proportional to $(\kappa_l - \kappa_u)$ that $\nu_m^{c,s}$ change sign upon exchanging the thermal conductivities. The further transformation $U_n \to -U_n$ which, as is evident from Eq. (24), amounts to reversing the direction of the z-axis, leaves then the system unchanged. We thus conclude that the problem is invariant under an exchange of the conductivities of the two plates. The same conclusion has been reached for the fully non-linear problem in Ref. 11.

Equations (33) and (34) constitute a homogeneous linear algebraic system of infinite order. The solvability condition determines lines $Ra = Ra(k)$, which express the stability boundaries of the different modes of the system.

V. ONE-TERM TRUNCATION

Since both u'_z and its derivative vanish at $z = \pm 1/2$, a standard theorem on the Fourier series guarantees that the coefficients U_n and $\tilde{U}_n$ decrease at least as fast as n^{-3}.[12,13] Therefore, it may be expected that a low-order truncation—in particular limited to $n = 1$—may already give a useful approximation. From Eqs. (33) and (34), the relevant equations are then

$$\left[1 - \alpha_1^c - \beta_1^c\mu_1^c - \frac{(\pi^2+k^2)^3}{Ra\,k^2}\right]\tilde{U}_1 - \beta_1^c\nu_1^c U_1 = 0, \quad (39)$$

$$\beta_1^s\nu_1^s\tilde{U}_1 + \left[1 + \beta_1^s\mu_1^s - 2\alpha_1^s - \frac{(4\pi^2+k^2)^3}{Ra\,k^2}\right]U_1 = 0. \quad (40)$$

Let us now consider a few special cases.

A. Constant plate temperature

This is the standard textbook case and can be recovered from the general expressions given in the Appendix by taking $\kappa_{l,u} \to \infty$. In this case, $\mu_1^{c,s} = \nu_1^{c,s} = 0$ and the threshold for the even modes is simply

$$k^2\,Ra_{even} = (\pi^2+k^2)^3\left[1 - \frac{16\pi^2 k}{(\pi^2+k^2)^2}\frac{\cosh^2 k/2}{\sinh k + k}\right]^{-1}, \quad (41)$$

while, for the odd modes,

$$k^2 Ra_{odd} = (4\pi^2+k^2)^3\left[1 - \frac{64\pi^2 k}{(4\pi^2+k^2)^2}\frac{\sinh^2 k/2}{\sinh k - k}\right]^{-1}. \quad (42)$$

The results for the even and odd modes are shown by the dashed lines in Figures 1 and 2, respectively; the symbols show the exact values given in Refs. 1 and 5. A comparison of the actual numerical values is given in Table I. It is seen that the even modes are affected by an error of about 0.5%, while the odd modes have a slightly larger error of the order of 1%. The minimum of the curve for the even modes occurs at $k \simeq 3.114$ rather than the exact value $k \simeq 3.117$, a difference of less than 0.1%. For the odd modes, the minimum occurs at the same k as for the exact calculation to the precision available for the latter.

B. Equal thermal conductivities

For equal thermal conductivities, $\kappa_l = \kappa_u = \kappa$, as seen from Eq. (38), $\nu_n^{c,s} = 0$ and again the even and odd modes are uncoupled. The threshold of the even modes is given by

$$k^2\,Ra_{even} = \frac{(\pi^2+k^2)^3}{1 - \alpha_1^c - \beta_1^c\mu_1^c}, \quad (43)$$

and that of the odd modes by

$$k^2\,Ra_{odd} = \frac{(4\pi^2+k^2)^3}{1 + \beta_1^s\mu_1^s - 2\alpha_1^s}. \quad (44)$$

Comparisons of these approximations with the exact results of Ref. 5 are given in Figure 1 for the even modes and Figure 2 for the odd modes for $\kappa = 0.5$, 1, and 2.

C. Insulated plates

For small k, the quantities appearing in Eq. (43) are asymptotic to

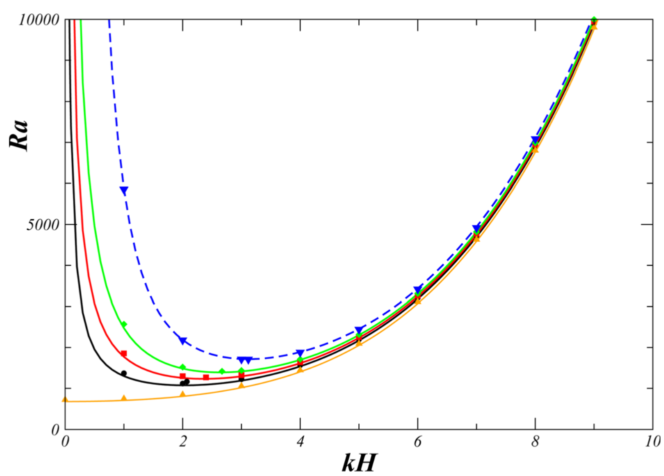

FIG. 1. (Color online) The Rayleigh number for the neutral stability of the even modes with equal plate thermal conductivities κ. In ascending order, the lines correspond to $\kappa = 0$ 0.5, 1, and 2. The topmost line, dashed, is for fixed plate temperature, i.e., $\kappa \to \infty$. The symbols show the exact results from Refs. 1 and 5.

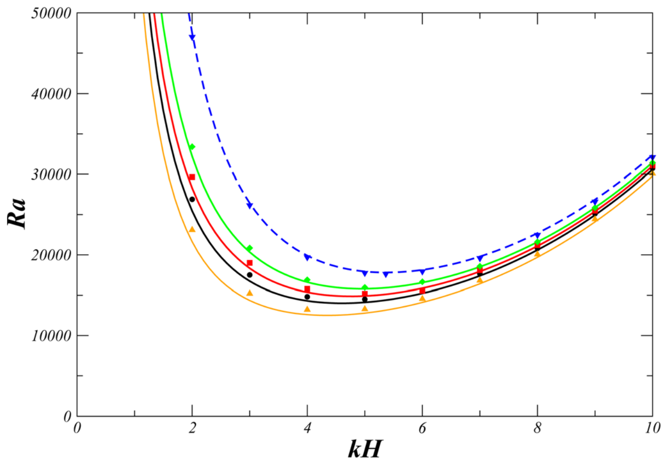

FIG. 2. (Color online) The Rayleigh number for the neutral stability of the odd modes with equal plate thermal conductivities κ. In ascending order, the lines correspond to $\kappa = 0, 0.5, 1$, and 2. The topmost line, dashed, is for fixed plate temperature, i.e., $\kappa \to \infty$. The symbols show the exact results from Refs. 1 and 5.

$$\alpha_1^c \simeq \frac{8}{\pi^2}, \quad \beta_1^c \simeq -\frac{1}{\pi^3}\left(\frac{4}{\pi^2} - \frac{1}{3}\right), \tag{45}$$

while

$$\mu_1^c \simeq \frac{\pi^5}{k} \frac{2 + (\kappa_l + \kappa_u)k}{(1 + \kappa_l \kappa_u)k + \kappa_l + \kappa_u}. \tag{46}$$

If the conductivities are equal and non-zero, the limit of this expression for $k \to 0$ is

$$\mu_1^c \simeq \frac{\pi^5}{k\kappa} \tag{47}$$

and

$$Ra_{even} \simeq \frac{\pi^4 \kappa}{k(4/\pi^2 - 1/3)} \tag{48}$$

tends to infinity for $k \to 0$. However, if $\kappa_l = \kappa_u = 0$ at the outset, we find $\mu_1^c \simeq 2\pi^5/k^2$ and

TABLE I. The Rayleigh number for neutral stability with fixed plate temperatures; the results labelled "exact" are from Refs. 1 and 5.

Wave number	Even modes			Odd modes		
K	Exact	Equation (41)	Error %	Exact	Equation (42)	Error %
1.0	5854.48	5870.35	0.27	163130.0	164719.66	0.97
2.0	2177.41	2184.89	0.34	47005.6	47482.89	1.0
3.0	1711.28	1718.38	0.41	26146.6	26420.84	1.0
3.114	—	1714.979	—	—	25273.26	—
3.117	1707.762	1714.98	0.42	—	25244.97	—
4.0	1879.26	1888.18	0.47	19684.6	19896.36	1.1
5.0	2439.32	2451.81	0.51	17731.5	17924.86	1.1
5.365	—	2757.87	—	17610.39	17802.60	1.1
6.0	3417.98	3435.81	0.52	17933.0	18128.09	1.1
7.0	4918.54	4943.52	0.51	19575.8	19784.58	1.1
8.0	7084.51	7118.33	0.48	22461.5	22692.43	1.0
9.0	10090.0	10133.76	0.43	26600.0	26859.50	0.97
10	14130.0	14190.08	0.43	32100.0	32398.15	0.92

$$Ra_{even} \to \frac{\pi^4}{8/\pi^2 - 2/3} \simeq 676.91 \qquad \text{for} \qquad k \to 0. \tag{49}$$

This problem can be solved exactly by a regular perturbation expansion in powers of k^2 and in this way one finds $Ra = 720$ as noted by Ref. 5, who also commented on the singular nature of the two limits $k \to 0$ and $\kappa \to 0$ for equal conductivities. The difference with Eq. (49) is about 6.4%. The perturbation expansion gives the leading order, $O(k^2)$, term of the velocity field

$$u_z' \propto \left(z^2 - \frac{1}{4}\right)^2, \tag{50}$$

from which we obtain the Fourier coefficients

$$\tilde{U}_n \propto \frac{(2n-1)^2\pi^2 - 12}{(2n-1)^5\pi^5}. \tag{51}$$

For large n, the decay rate is proportional to n^{-3} as expected.

No problem with the two limits affects the odd modes which, for equal conductivities and $k \to 0$, are found to be given by

$$Ra_{odd} \simeq \frac{1}{k^2}\frac{64\pi^6}{3 - 2\pi^2/15 - 6/\pi^2} \simeq \frac{57176.3}{k^2}. \tag{52}$$

No exact result with which this estimate can be compared appears to be available in the literature. A straightforward perturbation solution is given at the end of the Appendix. It is found that the numerical constant multiplying k^{-2} is 61528.9, which differs from Eq. (52) by about 7.6%.

The differences between our approximate theory and the exact results in the neighborhood of $k = 0$ are the largest ones that we have encountered. The explanation is likely that the term k^2 added to $(2n-1)^2\pi^2$ in equations such as Eq. (33) or to $4n^2\pi^2$ in Eq. (34) promotes a faster decay of the coefficients with increasing n. One may, therefore, have a greater confidence in the accuracy of the one-term truncation used here when the problem at hand centers on values of k that are not too small.

The lowest lines in Figures 1 and 2 show the results for insulated plates, $\kappa = 0$.

D. Short-wavelength perturbations

In the opposite limit of $k \to \infty$, for equal conductivities, we find

$$\begin{aligned} Ra_{even} &\simeq k^4\left(1 + \frac{3\pi^2}{k^2} + \frac{8\kappa + 5}{\kappa + 1}\frac{\pi^2}{k^3}\right), \\ Ra_{odd} &\simeq k^4\left(1 + \frac{12\pi^2}{k^2} + \frac{8\kappa + 5}{\kappa + 1}\frac{4\pi^2}{k^3}\right), \end{aligned} \tag{53}$$

both with an error of order k^{-4}. It is interesting to note that Jeffreys[14] proposed $Ra = k^4$ as the exact stability boundary for $\kappa_l = \kappa_u = 0$. As pointed out in Ref. 5, however, this result

TABLE II. The Rayleigh number for neutral stability of the two lowest modes when the temperature of one plate is fixed, while the other plate has a finite thermal conductivity.

	First mode				Second mode			
	Reference 6		Present		Reference 6		Present	
κ	*Ra*	*k*	*Ra*	*k*	*Ra*	*k*	*Ra*	*k*
0	1295.8	2.553	1262.6	2.525	15 278	4.91	15 143	4.888
0.01	1299.4	2.556	1266.4	2.529	—	—	15 165	4.891
0.03	1306.5	2.565	1273.9	2.538	—	—	15 206	4.897
0.1	1329.6	2.594	1298.5	2.567	15 467	4.94	15 343	4.917
0.3	1383.4	2.665	1356.2	2.638	—	—	15 665	4.968
1	1492.7	2.815	1475.0	2.793	16 382	5.11	16 342	5.080
3	1598.9	2.964	1592.3	2.951	—	—	17 040	5.211
10	1668.0	3.062	1670.1	3.055	17 380	5.31	17 518	5.307
100	1703.4	3.110	1710.1	3.108	—	—	17 772	5.358
∞	1707.8	3.117	1715.1	3.114	17 610	5.37	17 803	5.365

is incorrect as the associated fields fail to satisfy some of the boundary conditions.

E. One plate with fixed temperature

Nield[6] studied the case in which the temperature of the lower plate is kept fixed, so that $\kappa_l \to \infty$, while the upper plate has a finite thickness with its upper surface (not in contact with the fluid) at a fixed temperature. As already noted, he found that when the thickness of the upper plate is greater than that of the fluid layer, the plate behaves essentially as a semi-infinite solid, so that we can compare his results with ours.

In this case, the modes no longer have a definite parity, and the approximate characteristic equation is found by setting to zero the determinant of the systems (39) and (40). The lowest Rayleigh numbers for the first and second modes calculated by Nield are compared with those of the present approximation in Table II.

The picture that emerges from the comparison of Nield's exact result with ours is similar to that of Table I. The errors do not reach 3% in the worst case.

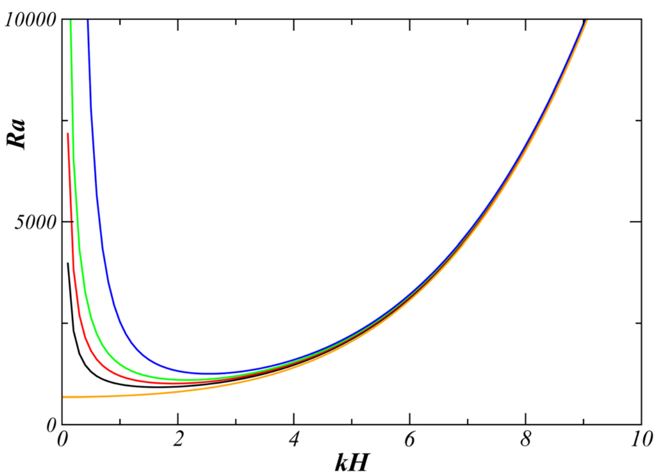

FIG. 3. (Color online) The Rayleigh number for the neutral stability of the lowest mode when one plate is insulated, while the thermal conductivity of the other one is, in ascending order, $\kappa = 0$, 0.5, 1, 2 and very large.

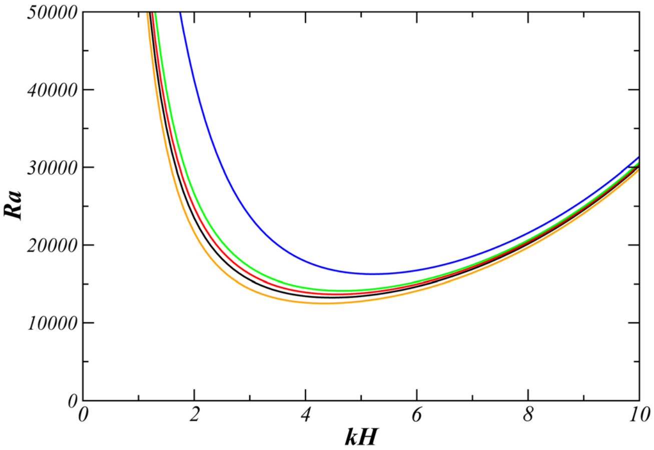

FIG. 4. (Color online) The Rayleigh number for the neutral stability of the second mode when one plate is insulated, while the thermal conductivity of the other one is, in ascending order, $\kappa = 0$, 0.5, 1, 2 and very large.

F. One plate insulated, the other conducting

One insulated and one thermally conducting plate is another situation in which the modes no longer have a definite parity, and the approximate characteristic equation is found by setting to zero the determinant of the systems (39) and (40).

As noted before the system is symmetric upon exchange of the thermal properties of the plates and, without loss of generality, we may take $\kappa_l = 0$, $\kappa_u = \kappa \geq 0$. Figure 3 shows the Rayleigh number for the neutral stability of the lowest mode and Figure 4 the Rayleigh number for the neutral stability of the next higher mode. In both figures, the lowest line is for $\kappa = 0$ and is the same as that shown in Figures 1 and 2, respectively.

VI. CONCLUSIONS

By solving the Rayleigh-Bénard linear marginal stability problem by a strategy somewhat different from the standard one, we have found relatively simple analytical approximations with an error of at most a few percent over the entire range of wave numbers and plate thermal conductivities.

The results confirm much that is already known: the stability threshold increases with the plate thermal conductivity and so does the critical wave number. The same trend is observed if both plates have the same thermal conductivity, or one has a fixed temperature (i.e., an infinite conductivity) and the other one a finite conductivity, or one is insulated and the other one conducting. Exact results for the last case do not seem to be in the literature.

While the present results refer to the linear threshold problem and are only approximate, their accuracy might make them useful as the starting point for weakly non-linear theories or to check other approximations or computer codes. Furthermore, they enable one to get a quick understanding of the system response to the plate thermal conductivities, the small- and large-wave number behavior and other features.

The same approach may be useful to study other situations such as plates of finite thickness, mixed stick-slip boundary conditions, convection in a porous medium, and possibly for other stability problems as well.

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APPENDIX: SOME ALGEBRAIC DETAILS

We give here some additional details on the calculations summarized in Sec. IV. A detailed derivation is provided in the supplementary material.[15]

Starting from Eq. (28), the velocity boundary conditions lead to

$$U_c = \frac{1}{32k^2}\frac{\sinh k - k}{\sinh k + k}W_c - \pi\frac{\sinh k/2}{\sinh k + k} \times \sum_{n=1}^{\infty}(2n-1)(-1)^n\tilde{U}_n, \tag{A1}$$

$$U_s = \frac{1}{32k^2}\frac{\sinh k + k}{\sinh k - k}W_s - \pi\frac{\cosh k/2}{\sinh k - k}\sum_{n=1}^{\infty}2n(-1)^n U_n, \tag{A2}$$

$$V_c = \frac{1}{4k^2}\frac{\sinh k - k\cosh k}{\sinh k + k}W_c + \frac{4\pi k\cosh k/2}{\sinh k + k} \times \sum_{n=1}^{\infty}(2n-1)(-1)^n\tilde{U}_n, \tag{A3}$$

$$V_s = \frac{1}{4k^2}\frac{\sinh k - k\cosh k}{\sinh k - k}W_s + \frac{4\pi k\sinh k/2}{\sinh k - k} \times \sum_{n=1}^{\infty}2n(-1)^n U_n. \tag{A4}$$

The remaining constants $W_{c,s}$ are found from the conditions on the temperature field, which is given by Eq. (19) the right-hand side of which is proportional to Eq. (26). The result of this calculation is

$$W_c = \sum_{n=1}^{\infty}(-1)^n[\nu_n^c U_n + \mu_n^c\tilde{U}_n], \tag{A5}$$

$$W_s = \sum_{n=1}^{\infty}(-1)^n[\mu_n^s U_n + \nu_n^s\tilde{U}_n], \tag{A6}$$

in which

$$\mu_n^c = (2n - 1)\pi\frac{(\kappa_l + \kappa_u)\sinh k/2 + 2\cosh k/2}{k[(\kappa_l\kappa_u + 1)\sinh k + (\kappa_l + \kappa_u)\cosh k]} \times [(2n-1)^2\pi^2 + k^2]^2, \tag{A7}$$

$$\mu_n^s = 2n\pi\frac{(\kappa_l + \kappa_u)\cosh k/2 + 2\sinh k/2}{k[(\kappa_l\kappa_u + 1)\sinh k + (\kappa_l + \kappa_u)\cosh k]} \times (4\pi^2 n^2 + k^2)^2, \tag{A8}$$

$$\nu_n^c = \frac{2n\pi(\kappa_l - \kappa_u)\sinh k/2}{k[(\kappa_l\kappa_u + 1)\sinh k + (\kappa_l + \kappa_u)\cosh k]}(4\pi^2 n^2 + k^2)^2, \tag{A9}$$

$$\nu_n^s = \frac{(2n-1)\pi(\kappa_l - \kappa_u)\cosh k/2}{k[(\kappa_l\kappa_u + 1)\sinh k + (\kappa_l + \kappa_u)\cosh k]} \times [(2n-1)^2\pi^2 + k^2]^2. \tag{A10}$$

Calculation of the integrals appearing in Eqs. (30) and (31) is facilitated by the relations (29) and one finds

$$W_c I_m^{H,c} + V_c I_m^{G,c} + U_c I_m^{F,c} = \frac{1}{2}\alpha_m^c\sum_{n=1}^{\infty}(-1)^{n+1}(2n-1)\tilde{U}_n - \frac{1}{2}\beta_m^c W_c, \tag{A11}$$

$$W_s I_m^{H,s} + V_s I_m^{G,s} + U_s I_m^{F,s} = \frac{1}{2}\alpha_m^s\sum_{n=1}^{\infty}(-1)^{n+1}2n\tilde{U}_n + \frac{1}{2}\beta_m^s W_s, \tag{A12}$$

in which

$$\alpha_m^c = (-1)^{m+1}\frac{16(2m-1)\pi^2 k}{[(2m-1)^2\pi^2 + k^2]^2}\frac{\cosh^2 k/2}{\sinh k + k}, \tag{A13}$$

$$\alpha_m^s = (-1)^{m+1}\frac{32m\pi^2 k}{(4m^2\pi^2 + k^2)^2}\frac{\sinh^2 k/2}{\sinh k - k}, \tag{A14}$$

$$\beta_m^c = \frac{(-1)^{m+1}(2m-1)\pi}{k^2[(2m-1)^2\pi^2 + k^2]^2} \times \left[\frac{k^2\sinh k/2 - k\cosh k/2 + \sinh k\cosh k/2}{\sinh k + k} - \frac{4k^2\cosh k/2}{(2m-1)^2\pi^2 + k^2}\right], \tag{A15}$$

$$\beta_m^s = \frac{2(-1)^{m+1}m\pi}{k^2(4m^2\pi^2 + k^2)^2} \times \left[\frac{k^2\cosh k/2 - k\sinh k/2 - \sinh k\sinh k/2}{\sinh k - k} + \frac{4k^2\sinh k/2}{4m^2\pi^2 + k^2}\right]. \tag{A16}$$

We conclude with a sketch of the solution procedure for the $k \to 0$ behavior of the lowest odd mode of the exact problem when the plates are insulated (i.e., $\kappa_{l,u} = 0$). We proceed perturbatively by setting $Ra = S/k^2 + O(1)$, $u_z' = u^0 + O(k^2)$, $T' = T^0 + O(k^2)$ in Eqs. (16) and (17). To lowest order, we are led to solving

$$\frac{\partial^4 u^0}{\partial z^4} - \frac{S}{Pr}T^0 = 0, \tag{A17}$$

$$\frac{\partial^2 T^0}{\partial z^2} + Pr\, u^0 = 0, \tag{A18}$$

subject to $u^0 = 0$, $\partial u^0/\partial z = 0$, $\partial T^0/\partial z = 0$ at $z = \pm 1/2$. By combining the two equations, we find

$$\frac{\partial^6 u^0}{\partial z^6} + Su^0 = 0 \tag{A19}$$

the real and odd solution of which may be written as

$$u^0 = A_0 \sin\left(S^{1/6} z\right) + A_1 \cos\left(\frac{S^{1/6}}{2} z\right) \sinh\left(\frac{\sqrt{3} S^{1/6}}{2} z\right) + A_2 \sin\left(\frac{S^{1/6}}{2} z\right) \cosh\left(\frac{\sqrt{3} S^{1/6}}{2} z\right). \tag{A20}$$

The temperature gradient is readily found from Eq. (A18) by integration. The requirements that it vanish at $z = 1/2$, together with the velocity and the velocity derivative, gives rise to a homogeneous system of 3 linear equations, the solvability condition of which is satisfied for a value of S somewhere in the range of $6.283185 < S^{1/6} < 6.283186$.

[1]S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Clarendon, Oxford, 1961) [reprinted by Dover, New York, 1981].
[2]P. G. Drazin and W. H. Reid, *Hydrodynamic Stability*, 2nd ed. (Cambridge University Press, Cambridge, UK, 2004).
[3]H. Bénard, "Les tourbillons cellulaires dans une nappe liquide," Rev. Gen. Sci. Pures Appl. **11**, 1261 (1900).
[4]Lord Rayleigh, "On convective currents in a horizontal layer of fluid when higher temperature is on the under side," Philos. Mag. **32**, 529 (1916).
[5]D. T. J. Hurle, E. Jakeman, and E. R. Pike, "On the solution of the Bénard problem with boundaries of finite conductivity," Proc. R. Soc. London **A296**, 469 (1967).
[6]D. A. Nield, "The Rayleigh-Jeffreys problem with boundary slab of finite conductivity," J. Fluid Mech. **32**, 393 (1968).
[7]C. Normand, Y. Pomeau, and M. G. Velarde, "Convective instability: A physicist's approach," Rev. Mod. Phys. **49**, 581 (1977).
[8]E. Bodenschatz, W. Pesch, and G. Ahlers, "Recent developments in Rayleigh-Bénard convection," Annu. Rev. Fluid Mech. **32**, 709 (2010).
[9]G. Ahlers, S. Grossmann, and D. Lohse, "Heat transfer and large scale dynamics in turbulent Rayleigh-Bénard convection," Rev. Mod. Phys. **81**, 503 (2009).
[10]D. Lohse and K.-Q. Xia, "Small-scale properties of turbulent Rayleigh-Bénard convection," Annu. Rev. Fluid Mech. **42**, 335 (2010).
[11]B. Holmedal, M. Tveitereid, and E. Palm, "Planform selection in Rayleigh-Bénard convection between finite slabs," J. Fluid Mech. **537**, 255 (2005).
[12]D. C. Champeney, *A Handbook of Fourier Theorems* (Cambridge University Press, Cambridge, UK, 1987).
[13]A. Prosperetti, *Advanced Mathematics for Applications* (Cambridge University Press, Cambridge, UK, 2011).
[14]H. Jeffreys, "The stability of a layer of fluid heated below," Philos. Mag. **2**, 833 (1926).
[15]See supplementary material at http://dx.doi.org/10.1063/1.3662466 for a detailed derivation of the results of the Appendix.